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Computation of Quasi-periodic Solutions of Forced Dissipative Systems II

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Quasi-periodicity with two periods we call bi-periodicity. We examine bi-periodic solutions of bi-periodically forced dissipative systems. The systems are described either by ordinary differential equations or by difference equations (iterated maps). When the ratio between the two periods is irrational or rational with many terms in its continued fraction expansion we can find the bi-periodic solution as a fixed point of a Poincaré map. The Poincaré map used to study bi-periodic solutions is a generalization of the Poincaré map used to study periodic solutions. We study a linear difference equation where the exact solution is known. Next we study a non-linear ordinary differential equation of the Duffing type, where we perform a continuation in the coefficient of the cubic term.

INTRODUCTION

We consider dynamical systems which are forced bi-periodically. The dynamics is assumed to be governed either by ordinary differential equations (ODEs) or difference equations (also called iterated maps (IMs)); thus

$$dx/dt = \dot{x} = f(t, t, x), \qquad t \geqslant 0, \quad x \in R^n, \quad n \geqslant 1$$

or

$$x(t+1) = \acute{x} = f(t, t, x), \qquad t = 0, 1, 2,..., \quad x \in R^n, \quad n \geqslant 1$$

where f is known, and

$$f: R_+ \times R_+ \times R^n \to R^n$$

$$f(\theta + T, \theta', x) = f(\theta, \theta', x)$$

$$f(\theta, \theta' + T', x) = f(\theta, \theta', x)$$

for all θ, θ' and x; T and T' are known periods. Thus f is bi-periodic. We assume f to be smooth. We assume the system to be dissipative, and to have a bi-periodic orbit; thus

$$x = b(t, t), \qquad 0 \leqslant t$$

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where

$$b: R_+ \times R_+ \to R^n$$
$$b(\theta + T, \theta') = b(\theta, \theta' + T') = b(\theta, \theta') \qquad \text{for all } \theta, \theta'$$

is a solution of the governing equations.

We note, that $b(0, \tau)$ is a periodic function with period T'. We define a Poincaré map P which has $b(0, 0)$ as a fixed point [6, 10]. In order to obtain $b(0, \tau)$ we observe $b(t, t)$ at times $t = 0, T, 2T, 3T,...$ and take $\tau = t$ modulo T'. Among all these points it turns out, that we only need a subset with τ values close to 0 or T'. That subset of points must have observation times close to the second period T'. These times can be determined from the continued fraction expansion of the ratio T/T', the so-called winding number. The stability of the fixed point defined in this way, can be determined from the eigenvalues of the derivative of the Poincaré map DP in that fixed point.

The theory of linear difference equations is similar to the theory of linear ordinary differential equations. This is demonstrated in [2]. We stress this similarity by denoting the discrete times 0, 1, 2, 3,... with t, as well as by introducing the symbol (́): $\acute{x}$ denotes the x vector one time step later.

Numerical solution methods for ODEs are based on difference equations [7]. Thus an ODE-solver transforms the dynamical system continuous in time into a dynamical system discrete in time.

The literature on bi-periodically forced ODEs is large. In [1, 3, 10] an approximation to the bi-periodic solution is performed using a generalized Fourier series. There the problem of small denominators arise. It turns out that our method does not have small denominators. In [11] perturbation methods are applied to multifrequency excitations of ODEs. On the other hand, the literature on bi-periodically forced iterated maps is sparse. In [12] a linear system of ODEs forced with Dirac delta functions have been reformulated as an iterated map.

The method discussed here was first announced in [8]. However, in that paper the following two questions were only vaguely answered:

(i) When can we define a Poincaré map?

(ii) How accurately can we compute the fixed point of the Poincaré map?

The answer to (i) is, that a Poincaré map can be defined when the ratio T/T' is irrational, or rational with a long continued fraction expansion. In order to answer (ii) we consider systems where the exact solution is known, and we use the algorithms in [4].

Bi-periodic solutions of both periodically forced systems of ODEs and IMs have been reported in [9]. The difficulty in these cases is T' is unknown. Bi-periodic solutions of autonomous ODEs were also studied in [9]; in that case both T and T' are unknown.

With two examples we demonstrate the application of the theory; an IM with known solution, and an ODE where the solution is known for one value of the parameter. This paper is computational in spirit, and our statements are based on numerical evidence.

The Method

The system of ordinary differential equations

$$\dot{x} = f(t, t, x), \qquad x(0) = x_0, \qquad x \in R^n, \quad 0 \leqslant t$$

defines an orbit $O(x_0)$ described by the solution curve $\phi(t; x_0)$, $0 \leqslant t$, where $\phi(0; x_0) = x_0$. Similarly the system of iterated maps

$$\dot{x} = f(t, t, x), \qquad x(0) = x_0, \qquad x \in R^n, \quad t = 0, 1, 2, \ldots$$

defines an orbit $O(x_0)$ starting in x_0.

The right-hand sides are bi-periodic functions, thus

$$f\colon R_+ \times R_+ \times R^n \to R^n,$$

$$f(\theta + T, \theta', x) = f(\theta, \theta' + T', x) = f(\theta, \theta', x) \qquad \text{for all } \theta, \theta'.$$

We assume $0 < T < T'$. This can be assumed without loss of generality in the ODEs. However, for the IMs, T must be an integer and we may then take kT', an integer multiple of T', for the second period to satisfy $0 < T < kT'$.

We assume that the systems have a bi-periodic orbit described by $b(t, t)$, $b(0, 0) = x_b$. Thus

$$b\colon R_+ \times R_+ \to R^n,$$

$$\phi(t; x_b) = b(t, t) \qquad \text{with} \quad \phi(0; x_b) = b(0, 0) = x_b$$

$$b(\theta + T, \theta') = b(\theta, \theta' + T') = b(\theta, \theta'), \qquad 0 < T < T' \quad \text{for all } \theta, \theta'.$$

For a moment, let us assume we know the solution x_b. If we look on the solution ϕ at times T, $2T$, $3T$,..., jT,... we have

$$\phi(jT; x_b) = b(jT, jT) = b(0, jT) = b(0, j \cdot (T/T') \cdot T').$$

We define the winding number $w = T/T'$, $0 < w < 1$. Then

$$\phi(jT; x_b) = b(0, jw \cdot T').$$

Since $b(0, \cdot)$ is periodic with period T', then we instead of writing jw can write τ_j, where τ_j is jw reduced to some interval of length one; we have used the interval $[-\frac{1}{2}, \frac{1}{2}[$. If we let ROUND($jw$) be the integer closest to jw, then we have

$$\tau_j = jw - \text{ROUND}(jw)$$

and so

$$\phi(jT; x_b) = b(0, \tau_j \cdot T').$$

Let J be a set of integers (which we determine in the next section), such that τ_j, $j \in J$ are in a small interval around zero. We interpolate between the points $\phi(jT; x_b)$ parameterized by τ_j, where $j \in J$. We can then evaluate the interpolating curve in $\tau = 0$. Within computing accuracy the interpolating curve is in the point x_b.

The procedure above can be applied to the situation where x_0 is in some neighbourhood of x_b. Interpolation between the points $\phi(jT; x_0)$, $j \in J$ yields the point $p \in R^n$. We define $q = p - x_0$, and denote it the residual vector.

We now define the residual map Q which to any input x_0 yields the output q, i.e.,

$$Q: R^n \to R^n, \qquad Q(x_0) = q.$$

The bi-periodic solution x_b is then a zero point of the residual map.

We also define the Poincaré map P:

$$P: R^n \to R^n, \qquad P(x_0) = p.$$

The bi-periodic solution x_b is a fixed point of P. The Poincaré map defined here is a generalization of the Poincaré map or return map used in the study of periodic solutions [10]. The linearized stability of the bi-periodic solution is equivalent to the linearized stability of the fixed point of P.

The set of points $\Sigma(x_0) = \{\phi(jT; x_0) \in R^n: j = 0, 1, 2,...\}$ is denoted the strobed orbit, and each point is denoted a strobe. We have $\Sigma(x_0) \subset O(x_0)$. On the bi-periodic solution $\Sigma(x_b)$ is a point set on a closed curve σ. If w is irrational, then $\Sigma(x_b)$ is dense on σ. If w is rational with a long continued fraction expansion, then the points $\Sigma(x_b)$ are sufficiently close for our use; σ may cross itself in double points and is therefore not invariant.

The Winding Number

We define the winding number as the ratio between the two periods

$$w = T/T', \qquad 0 < w < 1.$$

The continued fraction expansion of w is written [1, 5, 10]

$$w = [w_1, w_2, w_3,...] = 1 \Bigg/ \left(w_1 + \frac{1}{w_2 + \cdots} \right)$$

where w_i are positive numbers. When this expansion is truncated at w_k then we obtain the rational numbers

$$r_k/s_k = [w_1, w_2,..., w_k], \qquad k = 1, 2, 3,...,$$

which we denote the successive convergents to w [5]. The numbers can be computed recursively by

$$r_k = w_k \cdot r_{k-1} + r_{k-2}, \qquad k = 2, 3, \ldots,$$
$$s_k = w_k \cdot s_{k-1} + s_{k-2}, \qquad k = 2, 3, \ldots,$$

with $r_1 = 1$, $r_0 = 0$, $s_1 = w_1$, $s_0 = 1$. The sequence of successive convergents alternate about w and converges to w [10]. Thus the distances

$$d_k = s_k w - r_k, \qquad k = 1, 2, 3, \ldots$$

alternate about zero and converge to zero.

For any w we can determine a sequence of numbers n_l, $l = 1, 2, 3, \ldots$ such that

$$\delta_l < 0 < \delta_l + w$$
$$\delta_l = n_l \cdot w - l, \qquad l = 1, 2, 3, \ldots .$$

When d_k is positive for a chosen s_k, let us find an n_l for which $-\frac{1}{2} < \delta_l < 0$, and let m satisfy $0 < \delta_l + m d_k < \frac{1}{2}$. Then the set of integers

$$J = \{n_l, n_l + s_k, \ldots, n_l + m s_k\}$$

leads to a set of strobes $S \subset \Sigma(x_0)$,

$$S = \{\phi(jT; x_0) \in R^n : j \in J\}$$

which can be parameterized by τ_j, $j \in J$. The point on the interpolating curve at $\tau = 0$ we denote p, $p \in R^n$.

When d_k is negative for a chosen s_k, let us find an n_l for which $0 < \delta_l + w < \frac{1}{2}$, and let m satisfy $-\frac{1}{2} < \delta_l + w + m d_k < 0$. Then the set

$$J = \{n_l + 1, n_l + 1 + s_k, \ldots, n_l + 1 + m s_k\}$$

can be used to obtain a set of strobes.

If the continued fraction of w breaks off after a certain w_k, then $w = r_k/s_k$. We may either use $J = \{s_k\}$—and thus study the bi-periodic solution as a periodic solution with period $s_k T$—or we can determine a set J by the procedure above.

The computational work is essentially equal to the work in computing $\phi(jT; x_0)$ where $j = \max J$. If therefore $1 \ll w_k$ for a certain k, then $s_{k-1} \ll s_k$ and since the set J is based on s_k, the elements of J may become too large for practical computations. A large w_k means that the rational number r_k/s_k is very close to w. We want small values of w_k to have flexibility in the definition of J. Therefore the slower r_k/s_k converge to w, the easier we may find J. The slowest convergence of r_k/s_k will take place for the winding number $[1, 1, 1, \ldots] = 2/(\sqrt{5} - 1) =$ the golden mean inverse, where the successive convergents are ratios of the Fibonacci numbers 1, 1, 2, 3, 5, 8,... each being the sum of its two predecessors.

Numerical Details

The interpolation was done with cubic splines. We used the IMSL-routines ICSCCU to set up the interpolating curve, and ICSEVU to evaluate the curve at $\tau = 0$.

The accuracy of the values of Q was determined with the algorithm in [4]. We called x_0 a zero point of Q when the norm of $Q(x_0)$ was below that accuracy. Then the initial condition on the bi-periodic solution is $x_b = x_0$.

The zero of the residual map Q was found with Newton–Raphson's method. The derivative DQ was approximated with forward differences. The steplength in the difference was found with the algorithm in [4].

The stability of the fixed point x_b of P was determined from the eigenvalues of $DP(x_b) = (DQ + I)(x_b)$. When all eigenvalues are inside the unit circle in the complex plane, the fixed point is stable. The eigenvalues λ_i of DP depend on the values of (n_l, s_k, m). However, if $|\lambda_i| < 1$ for one set (n_l, s_k, m) then λ_i will be inside the unit circle for any other (n_l, s_k, m)-set. Similarly for $|\lambda_i| > 1$.

The governing system of equations may depend on a control parameter. Thus for the ODEs we have

$$\dot{x} = f(t, t, x; c), \qquad c \in R.$$

Then the zero point x_b of the residual map depends on c. The path of zeros is implicitly defined and can be followed from any given point on the path. This will be demonstrated in the example with Duffing's equation.

An Example with an Iterated Map

We shall study the linear difference equation

$$\acute{\acute{u}} + \acute{u} + \tfrac{3}{4}u = \cos(\sqrt{2}\,\pi t),$$

$$\acute{\acute{u}} = u(t+2), \qquad \acute{u} = u(t+1), \qquad u = u(t), \qquad t = 0, 1, 2, \ldots .$$

We use the substitution $x_1 = u$, $x_2 = \acute{u}$, and obtain the IM

$$\acute{x}_1 = x_2$$

$$\acute{x}_2 = -x_2 - \tfrac{3}{4}x_1 + \cos(\sqrt{2}\,\pi t).$$

The determinant of the jacobian of the right hand side is $\frac{3}{4}$, thus the system is dissipative.

We see, that $T = 1$ and $T' = \sqrt{2}$, thus the winding number is $w = 1/\sqrt{2}$ with the continued fraction expansion $w = [1, 2, 2, 2, \ldots]$. We choose $k = 6$ for which $r_k = 70$, $s_k = 99$, such that $d_k = 3.57133 \cdot 10^{-3}$. Then we fix $l = 29$ for which $n_l = 41$ and $\delta_l = -8.62197 \cdot 10^{-3}$. With $m = 4$ we have $J = \{41, 140, 239, 338, 437\}$. When the

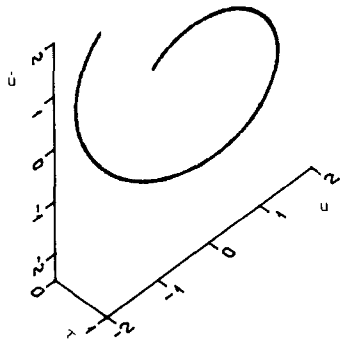

FIG. 1. Steady state solution of the difference equation $\ddot{u}+\dot{u}+\frac{3}{4}u=\cos(t\pi\sqrt{2})$. The initial condition at $t=0$ is $(u,\dot{u})=(-1.09085\ 67897\ 37628,\ 1.55572\ 71067\ 22602)$. The τ axis is $\tau=t/T'$ modulo 1, $t=0, 1, 2, \ldots, 500$, and $T'=\sqrt{2}$.

norm of the residual map Q was below 10^{-9} we stopped the Newton scheme. We found the zero point of Q to be

$$x_1=u(0)=-1.09085\ 67897\ 37628$$
$$x_2=\dot{u}(0)=\ \ 1.55572\ 71067\ 22602.$$

The eigenvalues of $DP=DQ+I$ in this point were inside the unit circle. Thus the bi-periodic solution is stable. A picture of the solution is seen in Fig. 1.

The difference equation can be solved exactly [2]. The solution is $u(t)=A(-\frac{1}{2}+i(\sqrt{2}/2))^t+B(-\frac{1}{2}-i(\sqrt{2}/2))^t+C\cos(\sqrt{2}\,\pi t)+D\sin(\sqrt{2}\,\pi t)$. Here $i=\sqrt{-1}$; A and B are arbitrary real constants, whereas C and D are the solutions of the 2×2 system

$$\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix},$$
$$\alpha=\cos(2\pi\sqrt{2})+\cos(\pi\sqrt{2})+\tfrac{3}{4},$$
$$\beta=\sin(2\pi\sqrt{2})+\sin(\pi\sqrt{2}).$$

The solution is bi-periodic when $A=B=0$, and because $|-\frac{1}{2}\pm i(\sqrt{2}/2)|<1$ the solution is stable. We have

$$u(0)=C \qquad\qquad = -1.09085\ 67916\ 34538$$
$$\dot{u}(0)=C\cos(\sqrt{2}\,\pi)+D\sin(\sqrt{2}\,\pi)=\ \ 1.55572\ 71093\ 23350,$$

which compares very well with our computations above.

An Example with an Ordinary Differential Equation

We shall study an equation of Duffing's type

$$\ddot{x} + 0.05\dot{x} + x + cx^3 = 0.3 \cos t + 1.5 \cos(0.115t),$$

where c is a control parameter. The case $c = 1$ was examined in [8] originally due to Chua and Ushida [3]. The case $c = 0$ can be solved exactly. The transient is damped, and the steady state solution is

$$x(t) = 6 \sin t + A \cos(0.115t) + B \sin(0.115t), \qquad t \geqslant 0.$$

This solution is bi-periodic, and A and B are the solutions of the 2×2 system

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.0 \end{bmatrix}$$

$$\alpha = 1 - 0.115^2$$

$$\beta = 0.05 \cdot 0.115.$$

We obtain

$$x(0) = \qquad A = 1.52005\ 17541\ 85237$$

$$\dot{x}(0) = 6 + 0.115B = 6.00101\ 86052\ 77247.$$

We use the substitution $y_1 = x$, $y_2 = \dot{x}$, and obtain the system

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -0.05y_2 - y_1 - cy_1^3 + 0.3 \cos t + 1.5 \cos(0.115t).$$

The divergence of the right-hand side is -0.05, thus the system is dissipative. The two periods are $T = 2\pi$ and $T' = 2\pi/0.115$. Therefore the winding number is $w = 0.115$, whose continued fraction expansion is [8, 1, 2, 3, 2]. The successive convergents are $\frac{1}{8}$, $\frac{1}{9}$, $\frac{3}{26}$, $\frac{10}{87}$ and $\frac{23}{200}$. From this we see, that the right-hand side is periodic with period $200 \cdot 2\pi$. We choose $k = 2$ and $l = 3$ such that $s_k = 9$ and $n_l = 17$. With $m = 3$ we then obtain $J = \{17, 26, 35, 44\}$.

We have performed a continuation in the parameter c. The initial values for the bi-periodic solutions are given in Table I. The corresponding strobed orbits can be seen in Fig. 2. The influence of the cubic non-linearity is seen to be very strong. In all cases the solutions are stable.

We used the IMSL-routine DVERK to solve the ODEs with a local error tolerance of 10^{-8}. When the norm of the residual map Q was below 10^{-7} the point was accepted as a zero point. The steplength used in the computation of DQ was $\sqrt{10^{-7}}$. The algorithm in [4] indicated, that $4 \cdot 10^{-7}$ was an optimal steplength. The reason may be, that the solution is very stable, i.e., all eigenvalues are close to zero.

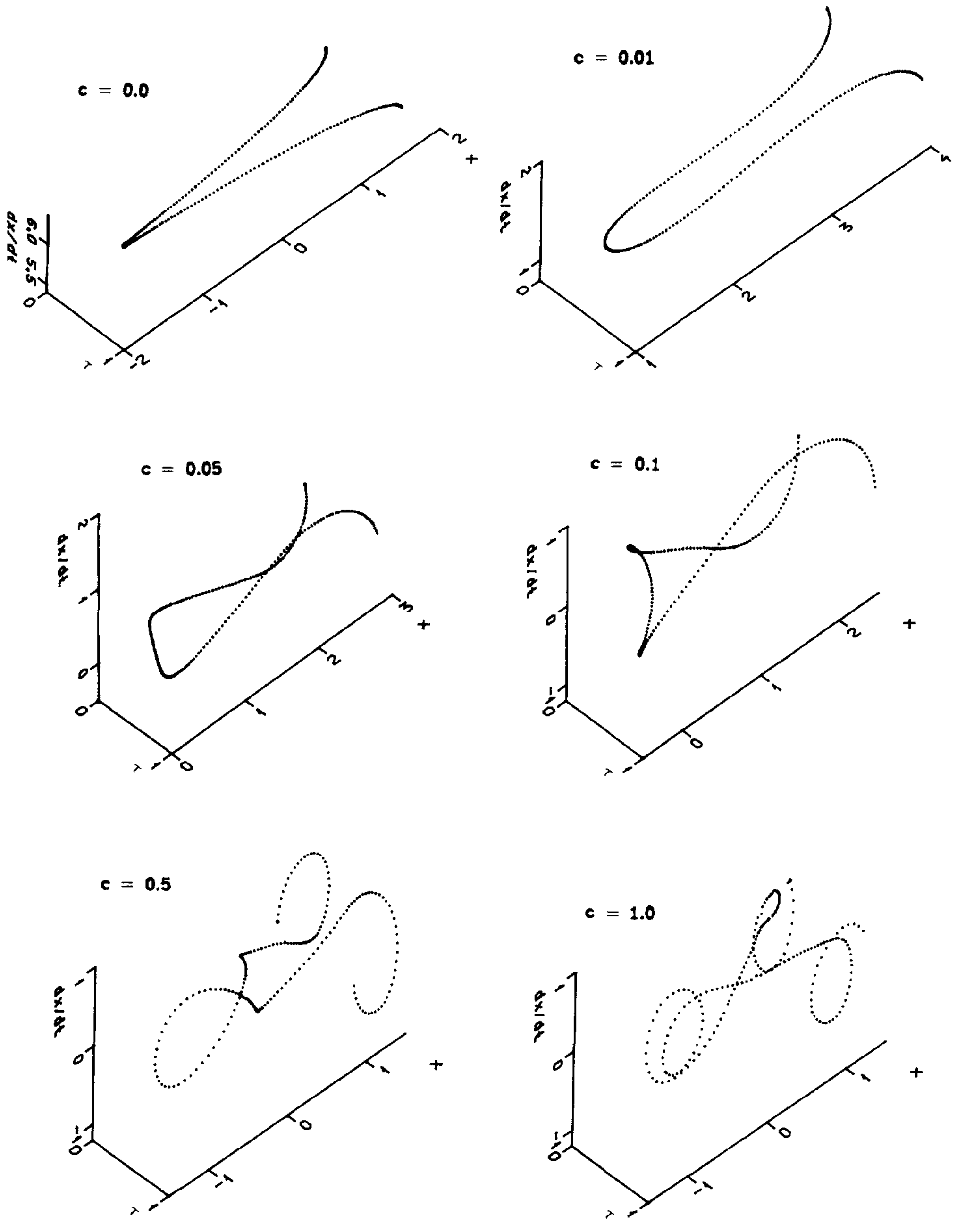

FIG. 2. Strobed orbits of the bi-periodic steady state solutions of $\ddot{x} + 0.05\dot{x} + x + cx^3 = 0.3\cos t + 1.5\cos(0.115t)$ for different values of c. The x and $\dot{x}$ axes have different scales, whereas the τ axis is $\tau = t/T'$ modulo 1, $t = 0, 2\pi, 2 \cdot 2\pi, \ldots, 200 \cdot 2\pi$, and $T' = 2\pi/0.115$.

TABLE I

Initial Conditions $(x, \dot{x})$ at $t=0$ for Bi-periodic Orbits

c	x	$\dot{x}$
0.0	1.5199 84759 91978 1	6.0010 18801 48837 8
0.01	3.9109 25574 43250 1	1.5296 40281 25913 2
0.05	2.8089 25069 83854 5	0.4044 34179 38561 66
0.1	2.4496 99465 33074 2	0.0847 64578 43474 747
0.5	0.8456 97528 20074 02	−0.0482 33694 28735 620
1.0	1.2018 82039 60392 9	0.2822 35981 73299 29

Conclusions

Given a bi-periodically forced system. When such a system has a bi-periodic solution, we have described a method to determine it. The method assumes the ratio of the two periods T and T' to be either irrational or rational with a long continued fraction expansion. If the continued fraction expansion is short the solution is better studied as a periodic solution. We have examined a system of IMs where we could compare our method with the exact solution. We have also examined a system of ODEs where other methods have been used earlier. In both cases with good results.

References

1. V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, New York/Heidelberg/Berlin, 1983).
2. C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw–Hill, New York, 1978).
3. L. O. Chua and A. Ushida, *IEEE Trans. Circuits Systems* **28** (1981), 953–971.
4. P. E. Gill, W. Murray, and M. H. Wright, *Practical Optimization*, (Academic Press, New York, 1981).
5. H. B. Griffiths and P. J. Hilton, *A Comprehensive Textbook of Classical Mathematics* (Springer-Verlag, New York/Heidelberg/Berlin, 1970).
6. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York/Berlin/Heidelberg/Tokyo, 1983).
7. P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations* (Wiley, New York/London/Sydney, 1962).
8. Chr. Kaas-Petersen, *J. Comput. Phys.* **58** (1985), 395–408.
9. Chr. Kaas-Petersen, *Physica D (Amsterdam)*, in press.
10. A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York/Heidelberg/Berlin, 1983).
11. A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations* (Wiley, New York, 1979).
12. A. Salat and J. Tataronis, in *Long-Time Prediction in Dynamics*, C. W. Horton, Jr., L. E. Reichl, and V. G. Szebehely, eds. (Wiley, New York, 1983).